**The Twin Prime Conjecture under modular analysis**

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**Abstract- The Twin Prime Conjecture is an unsolved problem in mathematics.** **The Twin Prime Conjecture posits that there are infinitely many prime numbers such that is also a prime number. The conjecture has not been proven or disproven yet. Proving this conjecture will be a groundbreaking discovery in mathematics. This study provides a very strong mathematical framework to analyze the conjecture.**

**1. Introduction**

The Twin Prime Conjecture is a long-standing and unsolved problem in number theory, positing that there are infinitely many pairs of prime numbers that differ by exactly two. Despite being a simple statement, the conjecture has eluded proof for centuries, making it one of the most famous unsolved problems in mathematics. Twin primes are closely related to the distribution of prime numbers and play a central role in understanding the overall structure of the primes.

This paper introduces a new approach to analyzing the Twin Prime Conjecture using modular analysis, a powerful tool in number theory. By considering the relationships between numbers modulo certain bases, we aim to explore new patterns and properties that may provide insight into the behavior of twin primes. Our goal is to build a framework that might lead to a potential proof of the conjecture or at least offer deeper understanding of the underlying structure of twin primes.

We begin by exploring foundational concepts such as the 2-3 Linear Summation Law and its relevance to prime number generation. We then apply modular arithmetic to investigate the conditions under which a number can be expressed as a sum of multiples of 2 and 3, and how this can help us identify twin primes. The application of the Chinese Remainder Theorem also offers a unique perspective on the relationship between numbers in the context of twin primes.

Ultimately, this paper aims to contribute a novel viewpoint on the Twin Prime Conjecture, using modular analysis as a means to uncover hidden patterns in the prime number sequence. The findings presented here could potentially lay the groundwork for a breakthrough in understanding twin primes and their distribution.

## **1.1 2-3 Linear Summation Law**

Every positive integer n, where n > 1, n can be expressed as a sum of multiples of 2 and 3. Or,

If > 0, If > 0,

Thus, if can be expressed as where and are non-negative integers, every number greater than n can be expressed as where and are non-negative integers.

The smallest number that can be expressed as a sum of two non-negative multiplicative of 2 and 3 is 2.

Thus, 2 and every number greater than 2 can be expressed as where m and n are non-negative integers.

## **1.2 Prime number**

Any number greater than 1 can be expressed as where and are non-negative integers. The smallest prime number is 2. Thus, any prime number can be expressed as where and are non-negative integers. and must be coprime to to even have a chance that is a prime. Although, m and n being coprime does not immediately imply that is a prime number. If, m and n are not coprime number, will be divisible by the .

(the fraction is rounded to the largest integer smaller than the fraction e.g. 3.4 is rounded to 3)

To to be prime, and and ... will all need to be coprime numbers, as they can be multiplied by 2 and 3 and their sum would result .

Let’s assume can be expressed as for multiple values of m and n where m and n are non-negative integers. Or,

Let the highest possible value for and be the lowest possible value for and be the second highest possible value of m and be the second lowest possible value for n and so one.

Thus,

And,

...

Or, We can generalize,

Because, =

( is an integer. not to be confused with the square root of -1)

Thus,

= = ... =

If is not coprime with where . is divisible by .

Because,

Thus, is only a prime number if and are coprime for every where “Assuming, is not an even number, it can be shown that if some and are coprime then there is also a which is divisible by .” Proving the statement is an exercise to the reader. In other words, is only a prime number where is not an even number if,

, , , .......

We know that,

So,

...

Thus, is only prime when,

, , ............,

Generalizing, must be prime when, 0 for every . Where where

If a can be expressed as where and are non-negative integers, then,

For and to both be prime,

, , ............,

Because, for to be prime and needs to be coprime for every v. where .

If, Then,

Thus, for to be prime and needs to be coprime for every v, where .

We know, for and to be prime these statements must be all true.

, , ............,

If are positive integers for any non-negative smaller or equal to . Then, four statements must be true.

They are-

1. is a prime number.
2. is also a prime number.

(Applied law of modular arithmetic)

1. If this statement is not true then either or is zero (Applied law of modular arithmetic). Which contradicts the statement.

From the statements we can observe that,

> 0 for every where, . If, and are prime numbers.

If is never smaller than 3 & is 1 for every non-negative integer v smaller than h, then, a and a+2 are always prime numbers.

We know,

We also know, Or (Applied laws of modular arithmetic) Zooguu X

## **1.3 Applied Chinese Remainder Theorem**

Let’s revise all the things that is proved. Let, be where is maximized, is minimized and they are non-negative numbers. Then, must be a prime number if,

, , , .......

We know,

.....

Or, We can generalize,

Thus, We can rewrite,

, , ............,

Let, b be a+2, Then,

Thus, Thus, must be a prime number when,

, , ............,

Or,

, , ............,

Thus, to a and a+2 both be prime,

, ,

,

..............,

Or, We can rewrite using the laws of modular arithmetic,

, , ,

..............,

( is the highest possible value for n. is the lowest possible value. Where )

If , is not greater than 3, then (Because 1 is always smaller than 3-1) for all v where implies is a prime number.

We know,

If,

Then, (Applied laws of modular arithmetic)

If, is not greater than 3, then, will at least be 9.

Because,

Thus, Or,

The Chinese remainder theorem states

If, .........

Where are coprime integers, There exists at least one solution for z.

We know,

If, where m is a non-negative integer , is maximized to the highest value possible, is greater or equal to 3 and n is manimized to the lowest value possible.

Then, must be prime when these statements are true,

) ) ) ) .......

Let be the smallest value possible 3. Then, We can rewrite the following statements as:

) = 1 ) = 4 ) = 0 ) = 1 .......

The chineese remainder theorem could be used to show that they are a infinite number of that satisfies the conditions.

If, .........

Where are coprime integers, there exists at least one solution for .

For i > 0, there are infinite solutions. Although, they do not directly result in a twin prime pairs. Because, For first congruences, if the solution is smaller or equal than 3 then a and are prime numbers. Where,

Let be the solution. if the solution for first congruences is larger than ,

For and to be prime,

) ) ) .... )

Remember,

) was derived from

So,

As, , There exists more congruences:

..... ).

We know that, if is 2 + 3, is maximized to the highest value possible and is minimized to the highest value ( & ),

Then, and must be a prime integer when,

, ,

,

..............,

Thus, for for any v from 0 to , there is exactly 2 remainders for which or must not be prime for the first congrunce ()

The forbidden 2 remainders are 0 and because, if is 0 then, is not greater than 0, so, must not be a prime. else if, is then is 0 which is definitely not greater than 0 thus must not be prime.

So, for every congruence () there are possible remainder values. As there are 2 forbidden remainder that makes non-prime. The possible remainder values if and has to be a prime integer. So, there are

(Because, is the count of possible remainder values. 2 of those are must be avoided if and must be a prime so, the valid count of remainder value is or ) numbers that satisfies the congruences to to be a prime number)

numbers that satisfy the congruence. If it can be proven that there will be at least one number in these numbers that is smaller or equal to , then, The Twin Prime Conjecture would be proved.

**1.4 Twin Prime Distribution**

In the congruences we have analyzed, Any number between to have to satisfy first k+1 congruences and the congruences are:

for (Because, is either 0,1 or 2 for . For any larger is negative).

Thus, for , a have to satisfy first k congruences to and to be prime.

As there are two forbidden remainders in the congruences and each congruence have n\_v possible remainders (forbidden remainders included), the chance of a number to satisfy the congruence is ( is the number of possible solutions and is the number of possible valid solutions where forbidden remainders are impossible). As can be expressed as and can be rewritten as , the fraction can be rewritten as For =3 and i=0, The term of the product in the numerator (n\_h+2i-2) is 1. Thus, the numerator can be rewritten without an extra value of i (i=0) as ∏\_(i=1)^k▒〖1+2i-2〗. Thus, the fraction can be represented as (∏\_(i=1)^k▒〖1+2i-2〗)/(∏\_(i=1)^k▒〖1+2i〗).

We know,

=

=

=

Thus, for any uniformly chosen random number has chance of being a valid solution and the solutions are uniformly distributed between numbers.

There is positive integer between 1 and 3k-1. Thus, having at least one valid solution between 1 and 3+2k is which is at least 1 for k ≥ 2 or there will always exist one solution satisfying first 2 or more congruences for example 4 is a solution for the first two congruences:

4≡1 (mod 3) (forbidden remainders are 0, 2)

4≡4 (mod 5) (forbidden remainders are 3, 2)

Thus 2\*4+9 and 2\*4+11 are prime numbers or 17 and 19 are prime numbers.

Thus, there must be at least one valid solution (in range) satisfying first k congruences for every k larger than 1. Let the solution be a. Then, and 2a+11 are always prime number or 2a+9 and 2a+11 are twin primes.

Thus, for every k>1, there is always one number a that satisfies first k congruences. Thus, 2a+9 and 2a+11 are prime number for any of these values of a. Although, it is not guaranteed that for different values of k there is unique values of .

**1.5 More results**

There are multiple results supporting this congruence-based twin prime checking. Please visit https://github.com/tm-ahad/twin\_prime\_checker for a practically implemented version of the congruence checker. Currently, it has been shown that all number p smaller than 1 million and are prime satisfies the first congruences and any other prime number or composite number does not satisfy all of the congruences.

The theorem also suggests if p > 9 and p is not divisible by 2 and p+6 have to be prime or is a sexy prime pair. P has to follow these congruences:

We know, for some m and n

Thus. Thus, the congruences are:

Examining the first congruence , we observe:

Thus, the first congruence can be rewritten as , meaning there is only one forbidden remainder for this congruence.

Let be the probability that a number follows all remaining congruences. The absolute probability of a number satisfying the first congruence is , leading to a probability of for a number being a sexy prime.

In contrast, for twin primes, there are two forbidden remainders for every congruence, including the first one. Since n is the probability of a number satisfying other congruences after the first one, the probability of a number being a twin prime is . Since sexy primes have only one forbidden remainder, There is two allowed modulos on the first congruence or there is a 2/3 chance that a random number will satisfy the first congruence. Thus, the chance that a random number satisfies all the congruences is while twin primes have the chance , we deduce: Thus, sexy primes are exactly twice as dense as twin primes, a well-known fact in number theory.

Applying the same argument to cousin primes, we observe that cousin primes are equally as dense as twin primes, another established mathematical fact. This theorem, therefore, not only confirms well-known prime distributions but also provides an exact predictive framework for different types of prime pairs.

The predictive accuracy of this theorem makes it a powerful tool for understanding prime number distributions. Future research should explore whether this modular approach can be extended to other fundamental problems in number theory, such as Goldbach's Conjecture and prime gaps of arbitrary size